1. (1 pt)
For which values of $x$ does
$$
\sum_{n=0}^{\infty} (n+4)!x^n
$$
converge? If the answer is a finite list of $x$-values, enter the list with commas. If the answer is an interval, use interval notation.

**SOLUTION:**

**SOLUTION**
The series is
$$
\sum_{n=0}^{\infty} (n+4)!x^n = 4! + (1 + 4)!x + (2 + 4)!x^2 + \cdots
$$
For $x = 0$, this certainly converges: $4! + 0 + 0 + \cdots$.
For all other $x$, we can apply the ratio test:
$$
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+5)!x^{n+1}}{(n+4)!x^n} \right| = \lim_{n \to \infty} \frac{n+5}{n} |x| = 0
$$
So for all $x \neq 0$, the series diverges. So the answer is simply 0.

**Answer(s) submitted:**
- 0

(correct)

**Correct Answers:**
- 0

2. (1 pt)
Find the interval of convergence for the power series
$$
\sum_{n=0}^{\infty} \frac{(-1)^n n^n x^{2n}}{(2n)!}
$$

Give your answer using interval notation. If you need to use $\infty$, type INF. If there is only one point in the interval of convergence, the interval notation is $[a]$. For example, if 0 is the only point in the interval of convergence, you would answer with $[0]$.

**SOLUTION:**

**SOLUTION**
The series certainly converges for $x = 0$. For $x \neq 0$, we can apply the ratio test:
$$
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} \frac{n+1}{2n+2} x^{2n+2}}{(-1)^n \frac{9}{2} x^{2n}} \right| = \lim_{n \to \infty} \frac{n+1}{9n} \frac{1}{x^2} = 0
$$

So the series converges for $\frac{1}{9}x^2 < 1$ and diverges for $\frac{1}{9}x^2 > 1$. That is, it converges for $-3 < x < 3$ and diverges for $x < -3$ and $x > 3$.

There are two cases where $\frac{1}{9}x^2 = 1$.
If $x = -3$, then the series simplifies to $\sum_{n=0}^{\infty} n$ which is not convergent.
If $x = 3$, then the series simplifies to $\sum_{n=0}^{\infty} (-1)^n n$ which is also not convergent.

So the interval of convergence is $(-3, 3)$.

**Answer(s) submitted:**
- $(-3, 3)$

(correct)

**Correct Answers:**
- $(-3, 3)$

3. (1 pt)
Find the interval of convergence for the following power series:
$$
\sum_{n=1}^{\infty} \frac{x^n}{(n!)^9}
$$

The interval of convergence is: __________

**SOLUTION:**

**SOLUTION**
With $a_n = \frac{1}{(n!)^9}$,
$$
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{1}{((n+1)!)^9} \cdot \frac{(n!)^9}{1} \right| = \left( \frac{1}{n+1} \right)^9
$$
and
$$
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.
$$
The radius of convergence is therefore $R = r^{-1} = \infty$, and the series converges absolutely for all $x$. Thus, the interval of convergence is $(-\infty, \infty)$.

**Answer(s) submitted:**
- $(-\infty, \infty)$

(correct)

**Correct Answers:**
- $(-\infty, \infty)$
4. (1 pt)
Find the interval of convergence for the power series
\[
\sum_{n=2}^{\infty} \frac{(5n)!}{(n!)^5} (x + 2)^n
\]
Give your answer using interval notation. If you need to use \(\infty\), type INF. If there is only one point in the interval of convergence, the interval notation is \([a]\). For example, if 0 is the only point in the interval of convergence, you would answer with \([0]\).

**SOLUTION:**

The series certainly converges for \(x = -2\). For \(x \neq -2\), we can apply the ratio test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(5n+5)!}{(n+1)!^5} \frac{(x + 2)^{n+1}}{(x + 2)^n} = \lim_{n \to \infty} \frac{(5n+5)(5n+4)(5n+3)(5n+2)(5n+1)}{(n+1)^5} = \lim_{n \to \infty} \frac{5^5n^5 + \cdots}{n^5 + \cdots} |x + 2| = 0
\]

So the series converges for all \(x\). The interval of convergence is \((-\infty, \infty)\).

Answer(s) submitted:

- (-\infty, \infty)

(correct)

Correct Answers:

- (-\infty, \infty)

5. (1 pt)
Find the interval of convergence for the power series
\[
\sum_{n=3}^{\infty} \frac{(x + 9)^n}{(n \ln n)^3}
\]
Give your answer using interval notation. If you need to use \(\infty\), type INF. If there is only one point in the interval of convergence, the interval notation is \([a]\). For example, if 0 is the only point in the interval of convergence, you would answer with \([0]\).

**SOLUTION:**

The series certainly converges for \(x = -9\). For \(x \neq -9\), we can apply the ratio test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(x+9)^{n+1}}{(n+1)^3 \ln(n+1)} = \lim_{n \to \infty} \frac{(n \ln n)^3}{(n+1)^3 \ln(n+1)} \frac{(x+9)^{n+1}}{(x+9)^n} = \lim_{n \to \infty} \frac{1}{(1 + \frac{1}{n})^3 \ln(n+1)} = \lim_{n \to \infty} \frac{n}{n - \ln(n+1)} = \lim_{n \to \infty} \frac{n + 1}{n} = 1
\]

So the series converges for \(|x+9| < 1\) and diverges for \(|x+9| > 1\).

There are two cases where \(|x+9| = 1\).

If \(x = -10\), then the series simplifies to \(\sum_{n=5}^{\infty} \frac{(-1)^n}{n \ln n}\) which is convergent by the Leibniz Test.

If \(x = -8\), then the series simplifies to \(\sum_{n=5}^{\infty} \frac{1}{n \ln n}\) which is convergent the the Comparison Test with the convergent \(p\)-Series \(\sum_{n=5}^{\infty} \frac{1}{n^3}\).

So the interval of convergence is \([-10, -8]\).

Answer(s) submitted:

- [-10, -8]

(correct)

Correct Answers:

- [-10, -8]

6. (1 pt)
Expand the function \(f(x) = \frac{1}{1-10x}\) in a power series with center \(c = 0\).

\[
\frac{1}{1-10x} = \sum_{n=0}^{\infty} a_n x^n, \text{ where } a_n = \ldots
\]

What is the interval of convergence? Give your answer using interval notation. If you need to use \(\infty\), type INF. If there is only one point in the interval of convergence, the interval notation is \([a]\). For example, if 0 is the only point in the interval of convergence, you would answer with \([0]\).

The interval of convergence is

**SOLUTION:**

The geometric series tells us that
\[
\frac{1}{1 - r} = \sum_{n=0}^{\infty} r^n
\]

for \(|r| < 1\). So substituting 10x in for \(r\), we have
\[
\frac{1}{1 - 10x} = \sum_{n=0}^{\infty} (10x)^n = \sum_{n=0}^{\infty} 10^n x^n
\]

Since the geometric series equality holds exactly when \(|10x| < 1\), the equality we have holds exactly when \(|x| < \frac{1}{10}\).

So the interval of convergence is \((-\frac{1}{10}, \frac{1}{10})\).

Answer(s) submitted:
- \(10^{-n}\)
- \((-\frac{1}{10}, \frac{1}{10})\)

(correct)
Correct Answers:
- \(10^{-n}\)
- \((-\frac{1}{10}, \frac{1}{10})\)

7. (1 pt)

Expand the function \(f(x) = \frac{1}{11 - 3x}\) in a power series with center \(c = 0\).
\[
\frac{1}{11 - 3x} = \sum_{n=0}^{\infty} a_n x^n, \text{ where } a_n = \]

What is the interval of convergence? Give your answer using interval notation. If you need to use \(\infty\), type INF. If there is only one point in the interval of convergence, the interval notation is \([a]\). For example, if 0 is the only point in the interval of convergence, you would answer with \([0]\).

The interval of convergence is ______.

SOLUTION:

Divide out \(\frac{1}{11}\) so that the form is agrees with the geometric series formula.
\[
\frac{1}{11 - 3x} = \frac{1}{11} \cdot \frac{1}{1 - \frac{3}{11} x}
\]

The geometric series tells us that
\[
\frac{1}{1 - \frac{3}{11} x} = \sum_{n=0}^{\infty} \left(\frac{3}{11}\right)^n x^n
\]

for \(|x| < 1\). So substituting \(\frac{3}{11} x\) in for \(r\), we have
\[
\frac{1}{1 - \frac{3}{11} x} = \sum_{n=0}^{\infty} \left(\frac{3}{11}\right)^n 4x - 5x^2 + x^3 + 4x^4
\]

That is,
\[
\frac{1}{11 - 3x} = \frac{1}{11} \sum_{n=0}^{\infty} \left(\frac{3}{11}\right)^n x^n
\]

Since the geometric series equality holds exactly when \(|r| < 1\), the equality we have holds exactly when \(|\frac{3}{11} x| < 1\). That is, when \(|x| < \frac{3}{11}\).

So the interval of convergence is \((-\frac{3}{11}, \frac{3}{11})\).

Answer(s) submitted:
- \(\frac{3}{11}\)
- \((-\frac{3}{11}, \frac{3}{11})\)

(correct)
Correct Answers:
- \(\frac{3}{11}\)
- \((-\frac{3}{11}, \frac{3}{11})\)

8. (1 pt)

The following series converges to a rational function in \(x\) for \(x\) in a certain interval.
\[
F(x) = 1 + 4x - 5x^2 + x^3 + 4x^4 - 5x^5 + x^6 + 4x^7 - 5x^8 + \cdots
\]

What is the interval of convergence? Give your answer using interval notation. If you need to use \(\infty\), type INF. If there is only one point in the interval of convergence, the interval notation is \([a]\). For example, if 0 is the only point in the interval of convergence, you would answer with \([0]\).

The interval of convergence is ______.

SOLUTION:

SOLUTION
Let \(\{a_n\} = \{1, 4, -5, 1, 4, -5, 1, 4, -5, \ldots\}\). Then 0 \(\leq |a_n x^n| \leq 5|x^n|\).
\[
\sum_{n=0}^{\infty} 5|x^n| \text{ is convergent for } |x| < 1 \text{ by the ratio test. So }
\sum_{n=0}^{\infty} a_n |x^n| \text{ is convergent by the Comparison Test (for } |x| < 1).\]

So \(\sum a_n x^n\) is absolutely convergent, and therefore convergent (for \(|x| < 1\)).

On the other hand, \(\sum a_n x^n\) is divergent for \(|x| \geq 1\), since the terms would not approach 0.

So the interval of convergence is \((-1, 1)\).

So let \(|x| < 1\). Since \(\sum a_n x^n\) is convergent, it must converge to the same value that the sequence of every third partial sum converges to. That is:
\[
-5x^5 + x^6 + 4x^7 - 5x^8 + \cdots = (1 + 4x - 5x^2) + (x^3 + 4x^4 - 5x^5) + (x^6 + 4x^7 - 5x^8) + \cdots
= (1 + 4x - 5x^2)(1 + 4x - 5x^2) + \cdots
= [1 + (4x - 5x^2)]
\]

Answer(s) submitted:
\begin{itemize}
  \item \( \frac{1+4x-5x^2}{1-(x^3)} \)
  \item \((-1,1)\)
\end{itemize}

(correct)

Correct Answers:
\begin{itemize}
  \item \( \frac{1+4x-5x^2}{1-x^3} \)
  \item \((-1,1)\)
\end{itemize}